REMARKS ON $K3$ SURFACES WITH NON-SYMPLECTIC AUTOMORPHISMS OF ORDER 7

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Abstract. In this note, we treat a pair of a $K3$ surface and a non-symplectic automorphism of order $7m$ ($m = 1$, 3 and 6) on it. We show that if the fixed locus of a non-symplectic automorphism order 7 is "special" then the pair is unique up to isomorphism. And we describe fixed loci of non-symplectic automorphisms of order 21 and 42.

Contents

1. Introduction

Let X be an algebraic $K3$ surface. In the following, we denote by S_X, T_X and ω_X the Néron-Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic 2-form on X, respectively. Let σ_I be an automorphism on X of finite order I. It is called *non-symplectic* if and only if it satisfies $\sigma_I^*\omega_X = \zeta_I \omega_X$ where ζ_I is a primitive I-th root of unity. Non-symplectic automorphisms have been studied by Nikulin who is a pioneer and several mathematicians.

It is known that the dimension of a moduli space of $K3$ surfaces with a non-symplectic automorphism of order I is $\operatorname{rk} T_X/\Phi(I) - 1$ if $I \neq 2$ or $\operatorname{rk} T_X - 2$ if $I = 2$ [5, Section 11], where Φ is the Euler function. Then there exists some cases such that the dimension of a moduli space of $K3$ surfaces with a non-symplectic automorphism is zero.

Problem 1.1. Let X_I be a $K3$ surface and σ_I a non-symplectic automorphism of order I on X_I. When is a pair $(X_I, \langle\sigma_I\rangle)$ unique up to isomorphism?

Vorontsov [16] announced some answers (without proofs) for the problem. Finally these were proved by Kondo, Oguiso and Zhang.

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Theorem 1.2. [7, Theorem] Assume that T_{X_I} is unimodular and σ_I acts trivially on S_{X_I}. If I =66, 44, 42, 36, 28 or 12 and $\Phi(I) = \operatorname{rk} T_{X_I}$ then there exists a unique (up to isomorphism) $K3$ surface with σ_I.

Here a lattice L is called unimodular if and only if $L = \mathrm{Hom}(L, \mathbb{Z})$, i.e. L is isomorphic to its dual lattice. If the transcendental lattice is not unimodular then the following theorem is important.

Theorem 1.3. [12, §2, §4] Assume that T_{X_I} is not unimodular and σ_I acts trivially on S_{X_I} and $\Phi(I) = \operatorname{rk} T_{X_I}$. If $I = 3, 5, 7, 11, 13, 19, 5^2, 3^2, 3^3$ then there exists a (unique) algebraic $K3$ surface X_I with $\operatorname{rk} T_X = \Phi(I)$.

In some of the above cases, it seems that an assumption about the action of σ_I on S_{X_I} is important. We can see some uniqueness theorems by changing assumptions on σ_I. An important assumption of Theorem 1.4 and Theorem 1.5 is the order of σ_I. We show uniqueness of $K3$ surfaces with σ_I from only I.

Theorem 1.4. [8, Main Theorem 1 and 2] Pairs $(X_{66}, \langle\sigma_{66}\rangle)$, $(X_{33}, \langle\sigma_{33}\rangle)$, $(X_{44}, \langle\sigma_{44}\rangle)$, $(X_{50}, \langle\sigma_{50}\rangle)$, $(X_{25}, \langle\sigma_{25}\rangle)$ and $(X_{40}, \langle\sigma_{40}\rangle)$ are unique up to isomorphism, respectively.

Recently the following is proved.

Theorem 1.5. [6] Pairs $(X_{21}, \langle\sigma_{21}\rangle)$ and $(X_{42}, \langle\sigma_{42}\rangle)$ are unique up to isomorphism, respectively.

We remark that these theorems do not assume that non-symplectic automorphisms act trivially on the Néron-Severi lattice. Indeed if I = 66, 44, 21 and 42 then σ_I acts trivially on S_{X_I}.

If $\Phi(I) < 12$ then the uniqueness of $(X_I, \langle\sigma_I\rangle)$ is not induced by only I. An important assumption is the fixed locus of σ_I, hence forms of fixed loci induce uniqueness.

Theorem 1.6. The followings hold by [10, Theorem 3, Theorem 4] [11, Main Theorem 4] [13, Theorem 1.5 (3)] :

(1) If $X_3^{\sigma_3}$ consists of only (smooth) rational curves and possibly some isolated points and contains at least 6 rational curves then a pair $(X_3, \langle\sigma_3\rangle)$ is unique up to isomorphism.
(2) If $X_2^{\sigma_2}$ consists of only (smooth) rational curves and contains at least 10 rational curves then a pair $(X_2, \langle\sigma_2\rangle)$ is unique up to isomorphism.
(3) If $X_5^{\sigma_5}$ contains no curves of genus ≥ 2, but contains at least 3 rational curves then a pair $(X_5, \langle\sigma_5\rangle)$ is unique up to isomorphism.
(4) Put $M := \{x \in H^2(X_{11}, \mathbb{Z}) | \sigma_{11}^*(x) = x\}$. A pair $(X_{11}, \langle\sigma_{11}\rangle)$ is unique up to isomorphism if and only if $M = U \oplus A_{10}$.

It is well known that if I is prime then $I \leq 19$. But these theorems miss the case of $I = 7$. Moreover Jang [6] does not determine fixed loci of automorphisms. The main purpose of this paper is to prove the following theorem:

Main Theorem. (1) If $X_7^{\sigma_7}$ consists of only (smooth) rational curves and some isolated points and contains at least 2 rational curves then a pair $(X_7, \langle\sigma_7\rangle)$ is unique up to isomorphism.
(2) The fixed locus of σ_{21} consists of exactly 11 isolated points and one $\mathbb{P}^1$.
(3) The fixed locus of σ_{42} consists of exactly 9 isolated points and one $\mathbb{P}^1$.

Remark 1.7. It is easy to see that if σ_I holds the theorem (2) or (3) then pairs $(X_{21}, \langle\sigma_{21}\rangle)$ and $(X_{42}, \langle\sigma_{42}\rangle)$ are unique up to isomorphism by Theorem 1.5

We know further results for uniqueness. See also [15].

Throughout this article we shall denote by A_m, D_n, E_l the negative-definite root lattice of type A_m, D_n, E_l respectively. We denote by U the even indefinite unimodular lattice of rank 2 and $U(m)$ the lattice whose bilinear form is the one on U multiplied by m.

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2. Preliminaries

In this section, we collect some basic results for non-symplectic automorphisms on a $K3$ surface. For the details, see [9] and [2], and so on.

Lemma 2.1. Let σ_I be a non-symplectic automorphism of order I on X_I. Then

(1) The eigen values of $\sigma_I^* \mid T_{X_I}$ are the primitive I-th roots of unity, hence $\sigma_I^* \mid T_{X_I} \otimes \mathbb{C}$ can be diagonalized as:

$$\begin{pmatrix} \zeta_I E_q & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \zeta_I^n E_q & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \zeta_I^{I-1} E_q \end{pmatrix},$$

where E_q is the identity matrix of size q and $1 \le n \le I-1$ is co-prime with I.

(2) Let P be an isolated fixed point of σ_I on X_I. Then σ_I^* can be written as

$$\begin{pmatrix} \zeta_I^i & 0 \\ 0 & \zeta_I^j \end{pmatrix} \qquad (i + j \equiv 1 \mod I)$$

under some appropriate local coordinates around P.

(3) Let C be an irreducible curve in $X_I^{\sigma_I}$ and Q a point on C. Then σ_I^* can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta_I \end{pmatrix}$$

under some appropriate local coordinates around Q. In particular, fixed curves are non-singular.

Lemma 2.1 (1) implies that $\Phi(I)$ divides $\operatorname{rk} T_X$ and Lemma 2.1 (2) and (3) imply that the fixed locus of σ_I is either empty or the disjoint union of non-singular curves and isolated points:

$$X_I^{\sigma_I} = \{p_1, \ldots, p_M\} \amalg C_1 \amalg \cdots \amalg C_N,$$

where p_i is an isolated fixed point and C_j is a non-singular curve.

The global Torelli Theorem gives the following.

Remark 2.2. [8, Lemma (1.6)] Let X be a $K3$ surface and g_i $(i = 1, 2)$ automorphisms of X such that $g_1^*|S_X = g_2^*|S_X$ and that $g_1^*\omega_X = g_2^*\omega_X$. Then $g_1 = g_2$ in Aut (X).

The Remark says that for study of non-symplectic automorphisms, the action on S_X is important. Hence the invariant lattice $S_{X_I}^{\sigma_I} := \{x \in S_{X_7} | \sigma_I^*(x) = x\}$ plays an essential role for the classification of non-symplectic automorphisms.

Proposition 2.3. [2, Theorem6.3] The fixed locus $X_7^{\sigma_7}$ is of the form

$$X_7^{\sigma_7} = \begin{cases} \{p_1, p_2, p_3\} \amalg E & \text{if } S_{X_7}^{\sigma_7} = U \oplus K_7, \\ \{p_1, p_2, p_3\} & \text{if } S_{X_7}^{\sigma_7} = U(7) \oplus K_7, \\ \{p_1, p_2, \dots, p_8\} \amalg E \amalg \mathbb{P}^1 & \text{if } S_{X_7}^{\sigma_7} = U \oplus E_8, \\ \{p_1, p_2, \dots, p_8\} \amalg \mathbb{P}^1 & \text{if } S_{X_7}^{\sigma_7} = U(7) \oplus E_8, \\ \{p_1, p_2, \dots, p_{13}\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } S_{X_7}^{\sigma_7} = U \oplus E_8 \oplus A_6. \end{cases}$$

Here E is a non-singular curve of genus 1 and K_7 is the even negative definite lattice given by Gram matrix $\begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$.

3. Uniqueness of $K3$ surfaces with a certain fixed locus

In this section, we treat a pair $(X_7, \langle\sigma_7\rangle)$ whose the fixed locus $X_7^{\sigma_7}$ consists of (smooth) rational curves and isolated points and contains at least 2 rational curves. We show that the pair $(X_7, \langle\sigma_7\rangle)$ is unique up to isomorphism.

Proposition 3.1. The automorphism σ_7 acts trivially on S_{X_7}.

Proof. Since $X_7^{\sigma_7}$ has at least 2 rational curves, $X_7^{\sigma_7} = \{p_1, p_2, \dots, p_{13}\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1$ and $S_{X_7}^{\sigma_7} = U \oplus E_8 \oplus A_6$ by Proposition 2.3. We know that $\operatorname{rk} T_{X_7} \geq 6$ by Lemma 2.1 (1) and $\operatorname{rk} S_{X_7} \geq 16$ since it contains the invariant lattice $S_{X_7}^{\sigma_7}$ which is of rank 16. This gives $\operatorname{rk} T_{X_7} \leq 6$ so that $\operatorname{rk} T_{X_7} = 6$ and $\operatorname{rk} S_{X_7} = 6$, hence S_{X_7} coincides with $S_{X_7}^{\sigma_7}$. This implies that the action of σ_7 is trivial on the S_{X_7}. □

The following Corollary follows from Proposition 3.1 and Proposition 2.3.

Corollary 3.2. $S_{X_7} = U \oplus E_8 \oplus A_6$, $T_{X_7} = U \oplus U \oplus K_7$ and the fixed locus σ_7 has 2 non-singular rational curves and 13 isolated points: $X_7^{\sigma_7} = \{p_1, p_2, \dots, p_{13}\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1$

We recall that the dimension of a moduli space of $K3$ surfaces with a non-symplectic automorphism of order 7 is $\operatorname{rk} T_{X_7}/\Phi(7) - 1$. In our case, its dimension is 0. Indeed we have the following.

Theorem 3.3. A pair $(X_7, \langle\sigma_7\rangle)$ is unique up to isomorphism.

Proof. It follows from Proposition 3.1 and Theorem 1.3. □

Example 3.4. [7, (7.5)] Put

$$X_{\mathrm{Ko}} : y^2 = x^3 + t^3 x + t^8, \ \sigma_{\mathrm{Ko}}(x, y, t) = (\zeta_7^3 x, \zeta_7 y, \zeta_7^2 t).$$

Then X_{Ko} is a $K3$ surface with $S_{X_{\mathrm{Ko}}} = U \oplus E_8 \oplus A_6$ and σ_{Ko} is a non-symplectic automorphism of order 7 acting trivially on $S_{X_{\mathrm{Ko}}}$.

Example 3.5. [12, §4] Put $X_{\mathrm{OZ}} : y^2 = x^3 + t^5 x + t^4$. Then X_{OZ} is a $K3$ surface with $S_{X_{\mathrm{OZ}}} = U \oplus E_8 \oplus A_6$ and a non-symplectic automorphism of order 7.

In [12], a non-symplectic automorphism of order 7 is not constructed. But $\varphi(x, y, t) = (\zeta_7^3 x, \zeta_7 y, \zeta_7^4 t)$ is a non-symplectic automorphism of order 7 on X_{OZ}.

Of course, it is easy to see that these examples are the same, by analysing the elliptic fibration.

4. The fixed locus of a non-symplectic automorphism of order 21

We describe the fixed locus of a non-symplectic automorphism of order 21. First we recall the following.

Proposition 4.1. [6, Theorem 2.1] A non-symplectic automorphism of order 21 σ_{21} acts trivially on $S_{X_{21}}$.

Lemma 4.2. The Euler characteristic of $X_{21}^{\sigma_{21}}$ is $3+\mathrm{tr}(\sigma_{21}^*|S_{X_{21}})=13$.

Proof. We apply the topological Lefschetz formula to the fixed locus $X_{21}^{\sigma_{21}}$: $\chi(X_{21}^{\sigma_{21}}) = 2+\mathrm{tr}(\sigma_{21}^*|S_{X_{21}})+\mathrm{tr}(\sigma_{42}^*|T_{X_{42}})$. By [9, Theorem 3.1], $\mathrm{tr}(\sigma_{21}^*|T_{X_{21}}) = \zeta_{21}+\zeta_{21}^2+\zeta_{21}^4+\zeta_{21}^5+\zeta_{21}^8+\zeta_{21}^{10}+\zeta_{21}^{11}+\zeta_{21}^{13}+\zeta_{21}^{16}+\zeta_{21}^{17}+\zeta_{21}^{19}+\zeta_{21}^{20} = -((1+\zeta_{21}^3+\zeta_{21}^6+\zeta_{21}^9+\zeta_{21}^{12}+\zeta_{21}^{15}+\zeta_{21}^{18})+(\zeta_{21}^7+\zeta_{21}^{14}) = -(0+(\zeta_3+\zeta_3^2) = -(0-1) = 1$. Since $\Phi(21)$=12, $\mathrm{rk}\, S_{X_{21}} = 10$. □

Lemma 4.3. Let $P_I^{i,j}$ be an isolated fixed point given by the local action $\begin{pmatrix}\zeta_I^i & 0\\ 0 & \zeta_I^j\end{pmatrix}$ and $m_I^{i,j}$ the number of $P_I^{i,j}$. Then we have

$$\begin{cases} m_{21}^{2,20}+m_{21}^{6,16}+m_{21}^{9,13} & \leq 4,\\ m_{21}^{3,19}+m_{21}^{5,17}+m_{21}^{10,12} & \leq 3,\\ m_{21}^{4,18}+m_{21}^{11,11} & \leq 1,\\ m_{21}^{2,20}+m_{21}^{5,17}+m_{21}^{8,14}+m_{21}^{11,11} & \leq 4.\end{cases}$$

Proof. Since $\sigma_{21}^3(P_{21}^{i,j})$ is a fixed point of σ_7, $P_{21}^{i,j}$ is mapped to $P_7^{i',j'}$ ($i\equiv i'$, $j\equiv j'$ mod 7). Thus

$$\begin{cases} P_{21}^{2,20}, P_{21}^{6,16}, P_{21}^{9,13} & \xmapsto{\sigma_{21}^3} P_7^{2,6},\\ P_{21}^{3,19}, P_{21}^{5,17}, P_{21}^{10,12} & \xmapsto{\sigma_{21}^3} P_7^{3,5},\\ P_{21}^{4,18}, P_{21}^{11,11} & \xmapsto{\sigma_{21}^3} P_7^{4,4},\\ P_{21}^{7,15}, P_{21}^{8,14} & \xmapsto{\sigma_{21}^3} Q_7\end{cases}$$

where Q_7 is a point on fixed curves of $\sigma_7=\sigma_{21}^3$. Since $\mathrm{rk}\, S_{X_{21}}=10$ and Proposition 4.1, we have

$$\begin{cases} m_{21}^{2,20}+m_{21}^{6,16}+m_{21}^{9,13} & \leq 4,\\ m_{21}^{3,19}+m_{21}^{5,17}+m_{21}^{10,12} & \leq 3,\\ m_{21}^{4,18}+m_{21}^{11,11} & \leq 1.\end{cases} \tag{4.1}$$

by [2, Theorem 2.4].

Moreover $\sigma_{21}^7(P_{21}^{i,j})$ is a fixed point of σ_3. If i or $j\equiv 0 \mod 3$ then $P_{21}^{i,j}$ is mapped to a point on a fixed curve of $\sigma_3=\sigma_{21}^7$ and if i and $j\not\equiv 0 \mod 3$ then $P_{21}^{i,j}$ is mapped to $P_3^{2,2}$. Since $\mathrm{rk}\, S_{X_{21}}=10$ and Proposition 4.1, we have

$$m_{21}^{2,20}+m_{21}^{5,17}+m_{21}^{8,14}+m_{21}^{11,11}\leq 4. \tag{4.2}$$

by [1, Theorem 2.2] and [14, Proposition 3.2] □

We apply the holomorphic Lefschetz formula ([3, page 542] and [4, page 567]) to $X_{21}^{\sigma_{21}}$:

$$\sum_{k=0}^{2} \mathrm{tr}(\sigma_{21}^*|H^k(X_{21},\mathcal{O}_{X_{21}})) = \sum_{i+j=22}^{M} a(P_{21}^{i,j}) + \sum_{l=1}^{N} b(C_l),$$

where $a(P_{21}^{i,j}) = 1/((1-\zeta_{21}^i)(1-\zeta_{21}^j))$ and $b(C_l) = (1-g(C_l))/(1-\zeta_{21}) - \zeta_{21}C_l^2/(1-\zeta_{21})^2$. Hence

$$1+\zeta_{21}^{20} = \sum_{i+j=22,\ 2\le i\le j} \frac{m_{21}^{i,j}}{(1-\zeta_{21}^i)(1-\zeta_{21}^j)} + \sum_{l=1}^{N} \frac{(1+\zeta_{21})(1-g(C_l))}{(1-\zeta_{21})^2}.$$

Then we have

$$\begin{cases} m_{21}^{6,16} = -\frac{m_{21}^{2,20}+m_{21}^{3,19}-m_{21}^{5,17}}{2} + 3\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{7,15} = 1-3m_{21}^{3,19} + 8\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{8,14} = 1-\frac{9m_{21}^{2,20}+3m_{21}^{3,19}+3m_{21}^{5,17}}{2} + 17\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{9,13} = 1-5m_{21}^{2,20}-m_{21}^{3,19}-2m_{21}^{5,17}+18\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{10,12} = 3+\frac{-15m_{21}^{2,20}+m_{21}^{3,19}+m_{21}^{5,17}}{2} - 3m_{21}^{4,18} + 21\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{11,11} = 1-3m_{21}^{2,20}-m_{21}^{4,18}+9\sum_{l=1}^{N}(1-g(C_l)). \end{cases} \tag{4.3}$$

Proposition 4.4. The fixed locus of σ_{21} consists of exactly 11 isolated points and one $\mathbb{P}^1$:

$$X_{21}^{\sigma_{21}} = \{P_{21}^{2,20}, P_{21}^{2,20}, P_{21}^{2,20}, P_{21}^{3,19}, P_{12}^{3,19}, P_{21}^{4,18}, P_{21}^{5,17}, P_{21}^{6,16}, P_{21}^{7,15}, P_{21}^{7,15}, P_{21}^{7,15}\}\amalg\mathbb{P}^1.$$

Proof. We remark inequalities in Lemma 4.3, equations (4.3) and $m_{21}^{i,j}$ is a non-negative integer.

If $m_{21}^{4,18}+m_{21}^{11,11} < 1$ then $m_{21}^{4,18} = m_{21}^{11,11} = 0$ and $m_{21}^{2,20} = 1/3+3\sum(1-g(C_l))$. This is a contradiction.

If $m_{21}^{2,20}+m_{21}^{6,16}+m_{21}^{9,13} = 3$ (resp. 2, 0) then $m_{21}^{3,19}+m_{21}^{5,17}-5\sum(1-g(C_l)) = -4/3$ (resp. $-2/3$, $2/3$). These are not integer. If $m_{21}^{2,20}+m_{21}^{6,16}+m_{21}^{9,13} = 1$ then $m_{21}^{5,17} = -m_{21}^{3,19}+5\sum(1-g(C_l))$ and $m_{21}^{8,19} = 1-4\sum(1-g(C_l))$. Hence $m_{21}^{5,17}$ or $m_{21}^{8,19}$ is negative. Thus $m_{21}^{2,20}+m_{21}^{6,16}+m_{21}^{9,13} = 4$.

If $m_{21}^{3,19}+m_{21}^{5,17}+m_{21}^{10,12} = 2$ (resp. 1) then $m_{21}^{4,18}-2\sum(1-g(C_l)) = -2/3$ (resp. $-1/3$). These are not integer. Assume $m_{21}^{3,19}+m_{21}^{5,17}+m_{21}^{10,12} = 0$. Since $m_{21}^{10,12} = 0$, we have $m_{21}^{5,17} = -2-m_{21}^{3,19}+5\sum(1-g(C_l))$. This contradicts for $m_{21}^{3,19} = m_{21}^{5,17} = 0$. Hence we have $m_{21}^{3,19}+m_{21}^{5,17}+m_{21}^{10,12} = 3$.

If $m_{21}^{2,20}+m_{21}^{5,17}+m_{21}^{8,14}+m_{21}^{11,11} = 2$ (resp. 1, 0) then $m_{21}^{10,12} = 5-5\sum(1-g(C_l))$ or $m_{21}^{7,15} = -5+2\sum(1-g(C_l))$ (resp. $-8+2\sum(1-g(C_l))$, $-11+2\sum(1-g(C_l))$) is negative. Assume $m_{21}^{2,20}+m_{21}^{5,17}+m_{21}^{8,14}+m_{21}^{11,11} = 3$. Then it is easy to see $M = \sum m_{21}^{i,j} = 10-2\sum(1-g(C_l))$. In particular $m_{21}^{2,20} = 3\sum(1-g(C_l))$, $m_{21}^{5,17} = -3+3\sum(1-g(C_l))$, $m_{21}^{8,14} = 4-4\sum(1-g(C_l))$ and $m_{21}^{11,11} = 2-2\sum(1-g(C_l))$. Since $m_{21}^{2,20}$, $m_{21}^{5,17}$, $m_{21}^{8,14}$ or $m_{21}^{11,11}$ is 0, we have $\sum(1-g(C_l)) = 1$ and $M = 8$. It follows from $\chi(X_{21}^{\sigma_{21}}) = M+\sum(2-2g(C_l))$ and Lemma 4.2 that $\mathrm{tr}(\sigma_{21}^*|S_{X_{21}}) = 7$. This is a contradiction for Proposition 4.1, hence $m_{21}^{2,20}+m_{21}^{5,17}+m_{21}^{8,14}+m_{21}^{11,11} = 4$.

In conclusion inequalities in Lemma 4.3 are equations. Moreover by (4.3), we have

$$\begin{cases} m_{21}^{2,20} & = 3\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{3,19} & = 2\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{4,18} & = -1+2\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{5,17} & = -2+3\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{6,16} & = -1+2\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{7,15} & = 1+2\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{8,14} & = 4-4\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{9,13} & = 5-5\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{10,12} & = 5-5\sum_{l=1}^{N}(1-g(C_l)), \\ m_{21}^{11,11} & = 2-2\sum_{l=1}^{N}(1-g(C_l)) \end{cases}$$

and $M = \sum m_{21}^{i,j} = 13 - 2\sum_{l=1}^{N}(1-g(C_l))$.

If $\sum_{l=1}^{N}(1-g(C_l)) \neq 1$ then $m_{21}^{4,18}$ or $m_{21}^{8,14}$ are negative. Thus we have $\sum_{l=1}^{N}(1-g(C_l)) = 1$, $M = 11$ and $\chi(X_{21}^{\sigma_{21}}) = M + \sum_{l=1}^{N}(2-2g(C_l)) = 11+2 = 13$. $\square$

5. The fixed locus of a non-symplectic automorphism of order 42

We describe the fixed locus of a non-symplectic automorphism of order 42. The following is a key in this section.

Proposition 5.1. [6, Corollary 2.6] A non-symplectic automorphism of order 42 σ_{42} acts trivially on $S_{X_{42}}$.

Lemma 5.2. The Euler characteristic of $X_{42}^{\sigma_{42}}$ is $1 + \mathrm{tr}(\sigma_{42}^*|S_{X_{42}}) = 11$.

Proof. We apply the topological Lefschetz formula to the fixed locus $X_{42}^{\sigma_{42}}$: $\chi(X_{42}^{\sigma_{42}}) = \sum_{i=0}^{4}(-1)^i \mathrm{tr}(\sigma_{42}^{*m}|H^i(X_{42},\mathbb{R})) = 1 - 0 + \mathrm{tr}(\sigma_{42}^*|S_{X_{42}}) + \mathrm{tr}(\sigma_{42}^*|T_{X_{42}}) - 0 + 1$. By [9, Theorem 3.1], $\mathrm{tr}(\sigma_{42}^*|T_{X_{42}}) = \zeta_{42}+\zeta_{42}^5+\zeta_{42}^{11}+\zeta_{42}^{13}+\zeta_{42}^{17}+\zeta_{42}^{19}+\zeta_{42}^{23}+\zeta_{42}^{25}+\zeta_{42}^{29}+\zeta_{42}^{31}+\zeta_{42}^{37}+\zeta_{42}^{41} = -((1+\zeta_{42}^2+\zeta_{42}^4+\cdots+\zeta_{42}^{40})+(\zeta_{42}^3+\zeta_{42}^7+\zeta_{42}^9+\zeta_{42}^{15}+\zeta_{42}^{21}+\zeta_{42}^{27}+\zeta_{42}^{33}+\zeta_{42}^{35}+\zeta_{42}^{39})) = -(0+(\zeta_{14}+\zeta_{14}^3+\zeta_{14}^5+\zeta_{14}^7+\zeta_{14}^9+\zeta_{14}^{11}+\zeta_{14}^{13})+(\zeta_6+\zeta_6^5)) = -(0+0+1) = -1$. Since $\Phi(21)$=12, $\mathrm{rk}\, S_{X_{21}} = 10$. $\square$

Lemma 5.3. The following inequalities and equations hold:

$$\begin{cases} m_{42}^{2,41} + m_{42}^{20,23} & \leq 3, \\ m_{42}^{3,40} + m_{42}^{19,24} & \leq 2, \\ m_{42}^{4,39} + m_{42}^{18,25} & \leq 1, \\ m_{42}^{5,38} + m_{42}^{17,26} & \leq 1, \\ m_{42}^{6,37} + m_{42}^{16,27} & \leq 1, \\ m_{42}^{7,36} + m_{42}^{15,28} & \leq 3, \end{cases}$$

and $m_{42}^{8,35} = m_{42}^{9,34} = m_{42}^{10,33} = m_{42}^{11,32} = m_{42}^{12,31} = m_{42}^{13,30} = m_{42}^{14,29} = 0$.

Proof. Since $\sigma_{42}^2(P_{42}^{i,j})$ is a fixed point of σ_{21}, $P_{42}^{i,j}$ is mapped to $P_{21}^{i',j'}$ ($i \equiv i'$, $j \equiv j'$ mod 21). It is easy to see these inequalities and equations by Theorem 4.4. $\square$

Proposition 5.4. The fixed locus of σ_{42} consists of exactly 9 isolated points and one $\mathbb{P}^1$:

$$X_{42}^{\sigma_{42}} = \{P_{42}^{2,41}, P_{42}^{2,41}, P_{42}^{2,41}, P_{42}^{3,40}, P_{42}^{3,40}, P_{42}^{4,39}, P_{42}^{5,38}, P_{42}^{6,37}, P_{42}^{7,36}\} \amalg \mathbb{P}^1.$$

Proof. We apply the holomorphic Lefschetz formula ([3, page 542] and [4, page 567]) to $X_{42}^{\sigma_{42}}$:

$$1+\zeta_{42}^{41} = \sum_{i+j=43,\ 2\le i\le j} \frac{m_{42}^{i,j}}{(1-\zeta_{42}^i)(1-\zeta_{42}^j)} + \sum_{l=1}^N \frac{(1+\zeta_{42})(1-g(C_l))}{(1-\zeta_{42})^2}.$$

Then we have

$m_{42}^{15,28} = 0,$

$m_{42}^{16,27} = 4m_{42}^{2,41} + 2m_{42}^{3,40} + 4m_{42}^{5,38} - 3m_{42}^{6,37} - m_{42}^{7,36} - 16\sum_{l=1}^N (1-g(C_l)),$

$m_{42}^{17,26} = -1 + 12m_{42}^{2,41} + 6m_{42}^{3,40} + 7m_{42}^{5,38} - 4m_{42}^{6,37} - 2m_{42}^{7,36} - 48\sum_{l=1}^N (1-g(C_l)),$

$m_{42}^{18,25} = -2 + 26m_{42}^{2,41} + 12m_{42}^{3,40} + m_{42}^{4,39} + 12m_{42}^{5,38} - 6m_{42}^{6,37} - 3m_{42}^{7,36} - 104\sum_{l=1}^N (1-g(C_l)),$

$m_{42}^{19,24} = 5 - 58m_{42}^{2,41} - 23m_{42}^{3,40} - 6m_{42}^{4,39} - 16m_{42}^{5,38} + 4m_{42}^{6,37} + m_{42}^{7,36} + 23\sum_{l=1}^N (1-g(C_l)),$

$m_{42}^{20,23} = 4 - 51m_{42}^{2,41} - 20m_{42}^{3,40} - 6m_{42}^{4,39} - 14m_{42}^{5,38} + 4m_{42}^{6,37} + m_{42}^{7,36} + 204\sum_{l=1}^N (1-g(C_l)),$

$m_{42}^{21,22} = 2 - 24m_{42}^{2,41} - 8m_{42}^{3,40} - 4m_{42}^{4,39} - 4m_{42}^{5,38} + 94\sum_{l=1}^N (1-g(C_l)).$

Moreover by Proposition 5.5, we have

$$\begin{cases} m_{42}^{2,41} & = 1 + 2\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{3,40} & = -2 + 4\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{4,39} & = -1 + 2\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{5,38} & = -1 + 2\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{6,37} & = -3 + 4\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{7,36} & = 1, \\ m_{42}^{16,27} & = 4 - 4\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{17,26} & = 2 - 2\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{18,25} & = 2 - 2\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{19,24} & = 4 - 4\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{20,23} & = 2 - 2\sum_{l=1}^N (1-g(C_l)), \\ m_{42}^{21,22} & = 2 - 2\sum_{l=1}^N (1-g(C_l)). \end{cases}$$

and $M = \sum m_{42}^{i,j} = 11 - 2\sum_{l=1}^N (1-g(C_l))$.

If $\sum_{l=1}^N(1-g(C_l)) \neq 1$ then $m_{42}^{4,39}$ or $m_{42}^{16,27}$ are negative. Thus $\sum_{l=1}^N(1-g(C_l)) = 1$ and $M = 9$. If the fixed locus $X_{21}^{\sigma_{21}}$ contains a non-singular curve then $X_{21}^{\sigma_{42}^2}$ also contain it. Thus $X_{42}^{\sigma_{42}}$ has at most one $\mathbb{P}^1$ by Proposition 4.4. □

Proposition 5.5. The following equations hold:

$$\begin{cases} m_{42}^{2,41} + m_{42}^{20,23} & = 3, \\ m_{42}^{3,40} + m_{42}^{19,24} & = 2, \\ m_{42}^{4,39} + m_{42}^{18,25} & = 1, \\ m_{42}^{5,38} + m_{42}^{17,26} & = 1, \\ m_{42}^{6,37} + m_{42}^{16,27} & = 1, \\ m_{42}^{7,36} + m_{42}^{15,28} & = 1. \end{cases}$$

Proof. We remark inequalities in Lemma 5.3 and $m_{42}^{i,j}$ is a non-negative integer.

If $m_{42}^{4,39} + m_{42}^{18,25} = 0$ (resp. $m_{42}^{5,38} + m_{42}^{17,26} = 0$) then $m_{42}^{16,27} = 2/3 - 14m_{42}^{2,41}/3 - 2m_{42}^{3,40} - 2m_{42}^{4,39}/3 - m_{42}^{6,37} + 56\sum(1-g(C_l))/3$ (resp. $m_{42}^{16,27} = 1/2 - 2m_{42}^{2,41} - m_{42}^{3,40} - -m_{42}^{6,37} + 8\sum(1-g(C_l))$). These are not integers, respectively.

If $m_{42}^{6,37} + m_{42}^{16,27} = 0$ then $m_{42}^{4,39} = -3/2 - m_{42}^{2,41} + 4\sum(1-g(C_l))$. This is not a integer.

If $m_{42}^{3,40} + m_{42}^{19,24} = 0$ ($m_{42}^{3,40} = m_{42}^{19,24} = 0$) then we have $m_{42}^{6,37} = -2 + 6m_{42}^{5,38}$ and $m_{42}^{7,36} = 3 - 8m_{42}^{5,38}$. $m_{42}^{6,37}$ or $m_{42}^{7,36}$ is negative. If $m_{42}^{3,40} + m_{42}^{19,24} = 1$ then $m_{42}^{6,37} = -3/2 + 4m_{42}^{2,41} + 6m_{42}^{5,38} - 16\sum(1-g(C_l))$. This is not a integer.

If $m_{42}^{7,36} + m_{42}^{15,28} = 3$ (resp. 2, 0) then $m_{42}^{5,38} = -1/4 - m_{42}^{2,41} + 4\sum(1-g(C_l))$ (resp. $= -1/8 - m_{42}^{2,41} + 4\sum(1-g(C_l)), = 1/8 - m_{42}^{2,41} + 4\sum(1-g(C_l))$). These are not integer.

If $m_{42}^{2,41} + m_{42}^{20,23} = 2$ (resp. 0) then $m_{42}^{5,38} = -1/ + 2\sum(1-g(C_l))$ (resp. $= 1/2 + 2\sum(1-g(C_l))$). These are not integer. If $m_{42}^{2,41} + m_{42}^{20,23} = 1$ then $m_{42}^{6,37} = -1 + 4\sum(1-g(C_l))$ and $m_{42}^{18,25} = 1 - 2\sum(1-g(C_l))$. $m_{42}^{6,37}$ or $m_{42}^{18,25}$ is negative. □

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